# Optimal Shock Isolation with Minimum Settling Time

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N88-13633

The computationally-determined limiting performance of shock isolation systems has been a useful tool in providing characteristics of optimal shock isolation. The limiting performance is defined as the minimum peak value of certain responses while other system responses are constrained. As is the case with most optimization problems, the "trajectory" in reaching the minimum performance index (peak response values) is unique, as is the minimum performance index itself. However, the responses of the system after the minimum performance index is achieved are not single-valued. This paper shows how unique isolator forces and corresponding responses can be chosen by superimposing a minimum settling time onto the limiting performance of the shock isolation system. Basically, this means that the system which has reached the peak value of the performance index is "settled" to rest in minimum time.

## INTRODUCTION

The limiting performance of a system is its absolute optimal response characteristics. It is computed by replacing those portions of a system being designed by active generic isolator forces. These isolator forces are then obtained so as to minimize a given performance index while typically satisfying bounding constraints on response variables or isolator force magnitudes. Since isolator forces are not restricted to represent any particular design elements during the optimization procedure, the resulting limiting performance response is optimal over all possible design configurations. No conditions are placed on the number or type of elements which are replaced by isolators; they may be active, pasive, or nonlinear. For the class of problems treated in this paper, the performance index and the constraints are linear combinations of system response variables and isolator forces. Also, the equations of motion are linear, so that it is possible to formulate the optimization procedure as a linear programming problem.

The limiting performance may be illustrated graphically by plotting the performance versus a constraint bound. If the performance index is chosen to minimize the maxmum response of a system subject to a prescribed constraint, the resulting tradeoff curve depicted in Fig. 1 gives the limiting performance of the system. The limiting performance charateristics are of considerable value to the mechanical system designer. First of all, they indicate from the design specifications alone whether a proposed design is feasible. Second, during the design cycle, they provide a measure of the success of the design configurations under consideration. Reference [1] describes limiting performance as applied shock isolation systems. Steady state systems [2], techniques for using general purpose

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structural analysis computer programs to generate the equations of motion for limiting performance studies [3], and the use of limiting performance characteristics in identifying the optimum design of suspension systems for rotating shafts [4] have been treated.

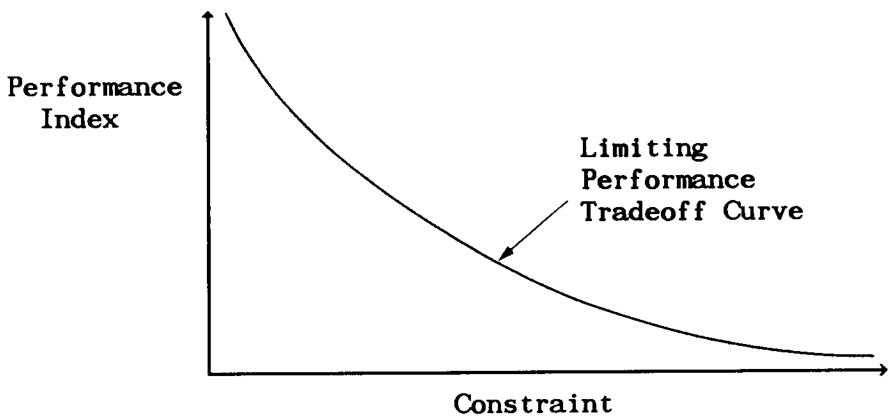



Fig. 1 Limiting Performance Characteristics

Although the limiting performance provides useful information, it has been noticed that the min-max norm of the limiting performance gives a unique solution only until the peak value of the performance index is achieved. A non-unique solution occurs after the peak value. For the tradeoff studies, the response after the peak is of little importance as long as the unique performance index is obtained. However, rapid settling of the disturbed system due to the external disturbance is often desired. Therefore, it is necessary to impose an additional measure of performance to obtain a unique solution after the peak value of the performance index is achieved. The response after the peak is selected to achieve the minimum settling time. Two different approaches can be used to achieve this goal [5]. This paper deals with the formulation using the performance index and its application to the shock isolation problems.

## PROBLEM STATEMENT

A linear vibrating system with n degrees of freedom subject to arbitrary external excitations $\underline{f}(t)$ and isolator forces $\underline{u}(t)$ is expressed in the first order system of differential equations

$$\underline{s}(t) = A\underline{s}(t) + B\underline{u}(t) + C\underline{f}(t) \tag{1}$$

where $\underline{s}(t)$ is an n-dimensional state vector and A, B, and C are n x n, n x nu and n x nf constant coefficient matrices. The quantities nu and nf are the number of isolator forces and excitations, respectively. Constraints are imposed on the dynamic system under study. The format of the constraints is

$$\underline{y}_L \leq Q_1\underline{s} + Q_2\underline{u} + Q_3\underline{f} \leq \underline{y}_U \qquad \text{for } t_o \leq t \leq t_f \tag{2}$$

where $\underline{y}_L$ and $\underline{y}_U$ are nc-dimensional lower and upper constraint vectors; $Q_1$, $Q_2$, and $Q_3$ are nc x n, nc x nu, and nc x nf constant coefficient matrices; and $t_o$ and $t_f$

are the given initial and final times.

The problem is to find an optimal isolator force $\underline{u}(t)$ which will transfer an initial state $\underline{s}(t_o) = \underline{s}_o$ to a desired final state $\underline{s}(t_f) = \underline{s}_f$ in the minimum time while extremizing a given performance index of the form

$$\text{Minimize } J = \{\max_{t_o \leq t \leq t_f} |\underline{p}_1^T\underline{s} + \underline{p}_2^T\underline{u} + \underline{p}_3^T\underline{f}|\} \tag{3}$$

where $\underline{p}_1$, $\underline{p}_2$, and $\underline{p}_3$ are given n, nu, and nf constant coefficient vectors. Since the min-max norm of the limiting performance gives a unique solution only until the peak value of the performance index is achieved, an additional measure of performance is desired to obtain a unique solution after the peak value. The resulting unique solution is referred to as limiting-performance/minimum-time (LP/MT) solution.

## LINEAR PROGRAMMING FORMULATION

To obtain the LP/MT solution, the performance index given in Eq. (3) is modified. Two sets of performance indices are considered. One set of them, referred to as the transient performance index, is given by

$$J_t = \max_{t_o \leq t < t_t} |\underline{p}_1^T\underline{s} + \underline{p}_2^T\underline{u} + \underline{p}_3^T\underline{f}| \tag{4}$$

where $t_t$ is the time limit for the transient period. The other set, referred to as the steady-state performance index is defined as

$$J_s = t_t \leq t \leq t_f \; |\underline{p}_1^T\underline{s} + \underline{p}_2^T\underline{u} + \underline{p}_3^T\underline{f}| \tag{5}$$

Now, the "global" performance index is defined by

$$J = J_t + J_s \tag{6}$$

Note that the vectors $\underline{p}_1$, $\underline{p}_2$, and $\underline{p}_3$ are not changed in Eqs. (4) and (5).

To place the optimization procedure into the standard linear programming form, the system in Eq. (1) is discretized using uniform time intervals to obtain a set of state difference equations

$$\underline{s}(k+1) = G\underline{s}(k) + H[B\underline{u}(k) + C\underline{f}(k)] \tag{7}$$

where $\underline{s}(k)$ = state vector at time $t = t_k$

$\underline{u}(k)$, $\underline{f}(k)$ = isolator force and external excitation vector at $t = t_k$, assumed to be constant over the interval $t_k \leq t < t_{k+1}$

$G = e^{Ah}$

$H = \int_o^h e^{A(h-\tau)} d\tau$

$$h = \text{time step} = t_{k+1} - t_k \quad (k=1, 2, \ldots, N-1)$$

The state vector, at any time $t = t_k$, can be expressed as a function of the initial state $\underline{s}(1)$ and the isolator force history $\underline{u}(1), \underline{u}(2), \ldots, \underline{u}(N-1)$ and the external excitation $\underline{f}(1), \underline{f}(2), \ldots, f(N-1)$. For $k = 1, 2, \ldots, N-1$

$$\underline{s}(k+1) = G^k \underline{s}(1) + \sum_{j=1}^{k-1} G^{k-j} H[B\underline{u}(j) + C\underline{f}(j)] + H[B\underline{u}(k) + C\underline{f}(k)] \tag{8}$$

The constraints in Eq. (2) are discretized similarly

$$\underline{y}_L(k) \leq Q_1 \underline{s}(k) + Q_2 \underline{u}(k) + Q_3 \underline{f}(k) \leq \underline{y}_U(k) \quad \text{for } k = 1,2,\ldots,N-1 \tag{9}$$

The objective functions of Eqs. (4) and (5), which reflect the min-max norm, are discretized and converted into a constraint set. Since $J_t$ is the maximum value of $|p_1^T \underline{s} + p_2^T \underline{u} + p_3^T \underline{f}|$ for $t_o \leq t < t_t$ and so is $J_s$ for $t_t \leq t \leq t_f$,

$$\begin{aligned} |p_1^T \underline{s}(k) + p_2^T \underline{u}(k) + p_3^T \underline{f}(k)| &\leq J_t \quad \text{for } t_o \leq t < t_t \\ |p_1^T \underline{s}(k) + p_2^T \underline{u}(k) + p_3^T \underline{f}(k)| &\leq J_s \quad \text{for } t_t \leq t \leq t_f \end{aligned} \tag{10}$$

To place this optimization problem into a standard linear programming form, define

$$\underline{z} = \begin{bmatrix} J_t \\ J_s \\ \underline{\underline{u}} \end{bmatrix} \tag{11}$$

where $\underline{\underline{u}} = [\ \underline{u}(1)^T\ \underline{u}(2)^T \ldots \underline{u}(N-1)^T\ ]^T$ (12)

and

$$\underline{c}^T = [\ 1 \quad 1 \quad 0 \ldots\ldots 0\ ] \tag{13}$$

Then the linear programming problem is to minimize

$$J = \underline{c}^T \underline{z} \tag{14}$$

subject to the constraints

$$H\underline{z} \leq \underline{b} \tag{15}$$

where H and $\underline{b}$ represent constraints of Eqs. (9) and (10).

The minimum time ($t_{min}$) is the smallest time which will make the global performance index of Eq. (6) stay within a desired value. Since the performance index can be computed for each iteration, an interpolation method such as the

secant method or simple bisection method [6] can be employed to find $t_{min}$ efficiently.

## NUMERICAL EXAMPLES

Example 1: A Single DOF System Subject to a Shock Velwave

A single degree-of-freedom (DOF) system composed of a mass m and supporting structure (Fig. 1) is subject to the horizontal shock velwave of Fig. 2.

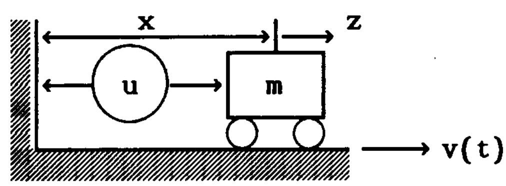
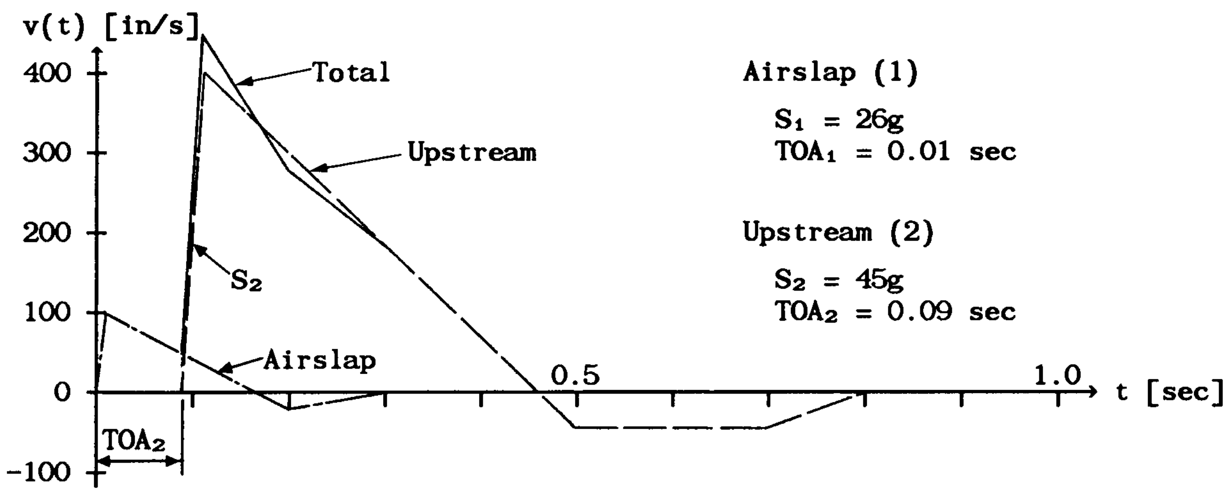



Fig. 1 A Single DOF Syatem




Note: Slope (S), Time of Arrival (TOA), and Total = 1 + 2

Fig. 2 Shock Velwave

Suppose the acceleration of the mass is to be limited to 15g (g = acceleration of gravity). The optimal isolator force $u^*(t)$ which minimizes the rattlespace between the mass and the supporting structure is desired.

The equation of motion is

$$\ddot{z} = u/m = U \tag{16}$$

The system is assumed to be at rest initially. The performance index to be minimized is

$$\max |z - y| \tag{17}$$

where

$$y = \int_{t_o}^{t_f} v(t)\, dt \tag{18}$$

and the constraint on the peak acceleration would be

$$|\ddot{z}| \leq 15g \tag{19}$$

Define a state vector

$$\underline{s} = [\ z \ \ \dot{z} \ \ y\ ]^T \tag{20}$$

The problem can now be transformed into the standard LP/MT isolator problem format described previously. The optimal isolator is sought, which will reduce the disturbed rattlespace to zero in minimum time while minimizing the performance index and satisfying the constraints. The resulting time responses are shown in Fig. 4. The performance index is 0.914 in and the minimum time is 0.17 sec.

Example 2: Two DOF Model of a Flexible Package Structure

A two mass model of a flexible package structure with a rigid base is shown in Fig. 5.

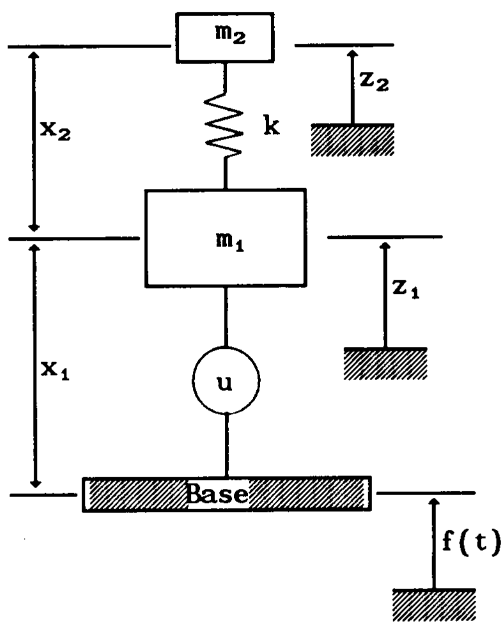



Fig. 5 Two DOF Flexible Package Model

The base is subject to external displacement which is described by

$$f(t) = 12t^2e^{-t} \text{ [in]} \tag{21}$$

The optimal isolator force is sought which will reduce the absolute displacement of

$m_2$ ($z_2$) within 5% of the peak value of the external disturbance f(t) in minimum time.

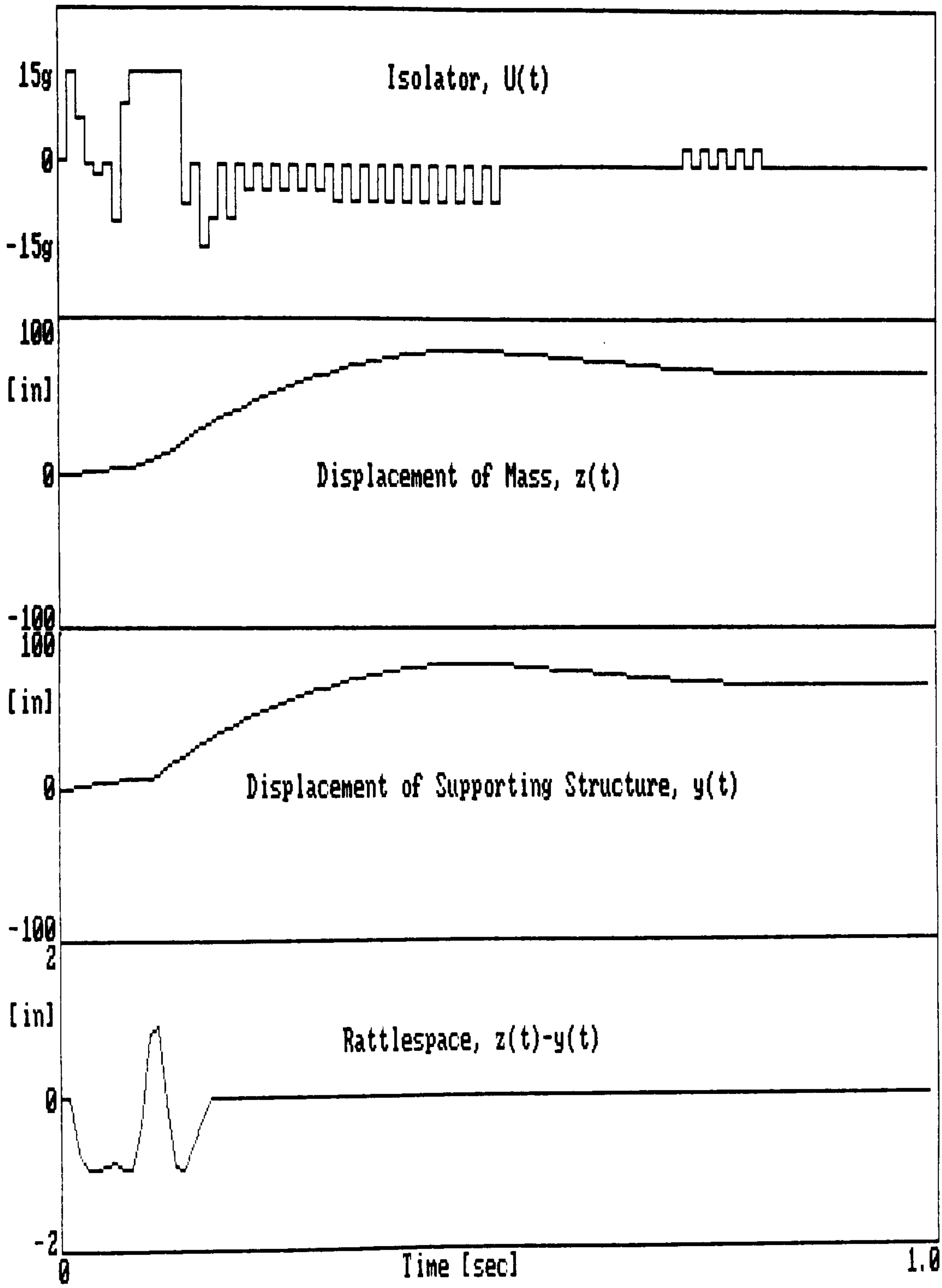



Fig. 4 Resulting Time Responses for Example 1

The performance index is to minimize

$$J = \max_t |z_2(t)| \tag{22}$$

while satisfying prescribed constraints

$$\begin{aligned} |U| &= |u/m_2| \leq U_{max} \\ |x_2| &= |z_2 - z_1| \leq X_{2max} \\ |x_1| &= |z_1 - f| \leq X_{1max} \end{aligned} \tag{23}$$

The equations of motion can be written as

$$\begin{aligned} \mu \ddot{z}_1 + \lambda(z_1 - z_2) &= U \\ \ddot{z}_2 + \lambda(z_1 - z_2) &= 0 \end{aligned} \tag{24}$$

where

$$\begin{aligned} \mu &= m_1/m_2 \\ \lambda &= k/m_2 \end{aligned} \tag{25}$$

Let

$$\underline{s} = [\ z_1\ \dot{z}_1\ z_2\ \dot{z}_2\ ]^T \tag{26}$$

Then the equations of motion, the performance index, and the constraints as given by Eqs. (24), (22), and (23), respectively, can be converted into the standard LP/MT format. Choose $U_{max}$ = 2g [in/sec$^2$], $X_{1max}$ = 2 [in], $X_{2max}$ = 1 [in], $\mu$ = 100, and $\lambda$ = 6.28 [rad/sec$^2$]. The solution for the optimal LP/MT isolator shows that the performance indices are $J_t$ = 5.729 [in] and $J_s$ = 0.099 [in], and the minimum time is $t_{min}$ = 12.0 [sec]. Figure 6 shows the time responses.

## CONCLUSIONS

The objective of this study was to show how unique isolator forces and corresponding responses could be chosen by superimposing a minimum settling time onto the limiting performance of the shock isolation systems. The limiting performance / minimum time characteristics were computed by linear programming. It was demonstrated that the superimposition of minimum settling time provided not only the value of the optimal performance index but also the minimum settling time which, in turn, gives unique solutions for shock isolation problems. The optimal LP/MT isolator characteristics can be used to check the feasibility of proposed design requirements and to measure the success of a given design during the design process by comparing the response of the designed system with that of LP/MT characteristics. Furthermore, the LP/MT charateristics would provide with a designer an insight to build a near optimal shock isolation system.

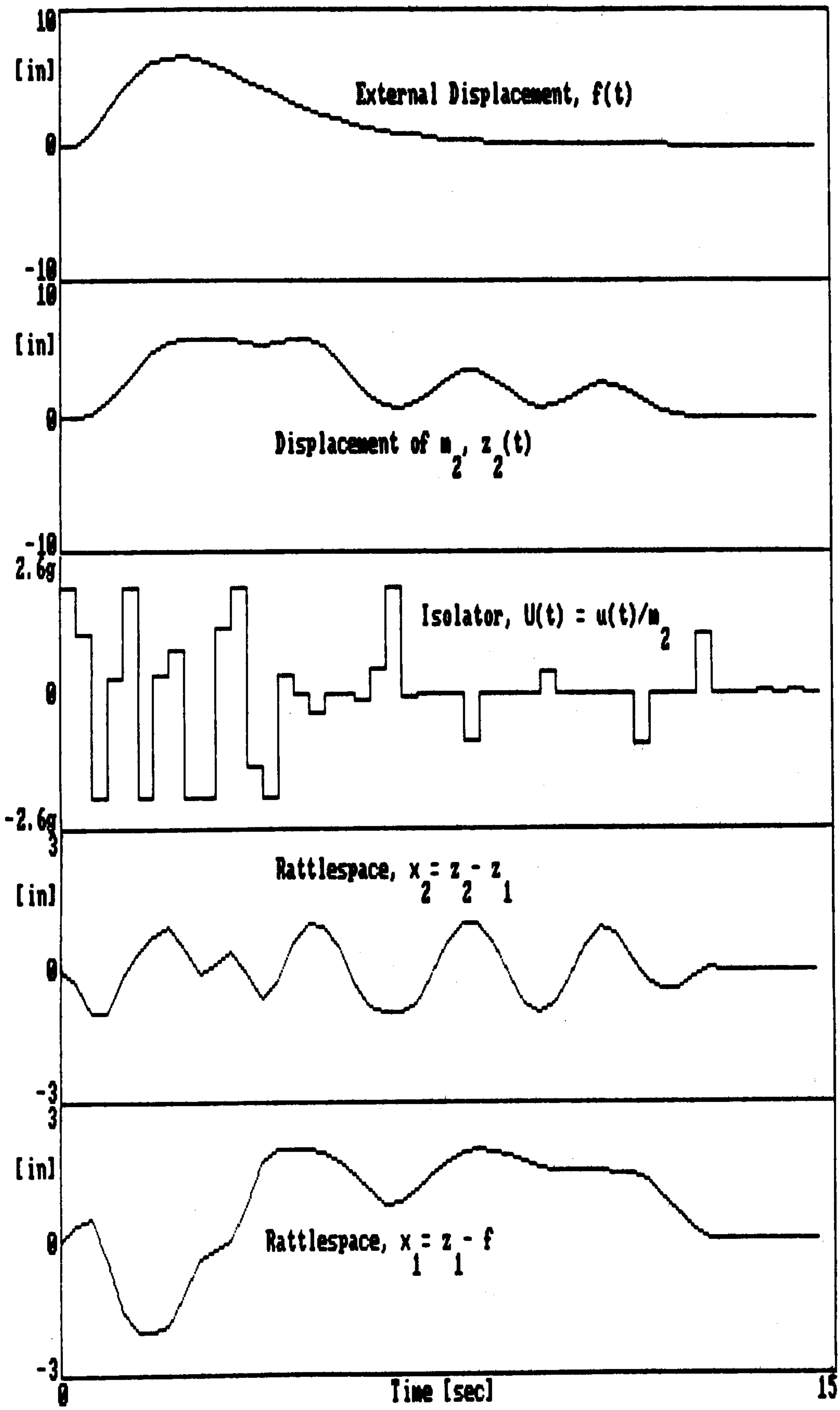



Fig. 6 Resulting Time Responses for Example 2

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